

DEL PEZZO SURFACES OVER FINITE FIELDS AND THEIR FROBENIUS TRACES

BARINDER BANWAIT, FRANCESC FITÉ, AND DANIEL LOUGHRAN

ABSTRACT. Let S be a smooth cubic surface over a finite field \mathbb{F}_q . It is known that $\#S(\mathbb{F}_q) = 1 + aq + q^2$ for some $a \in \{-2, -1, 0, 1, 2, 3, 4, 5, 7\}$. Serre has asked which values of a can arise for a given q . Building on special cases treated by Swinnerton-Dyer, we give a complete answer to this question. We also answer the analogous question for other del Pezzo surfaces, and consider the inverse Galois problem for del Pezzo surfaces over finite fields. Finally we give a corrected version of Manin's and Swinnerton-Dyer's tables on cubic surfaces over finite fields.

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1. INTRODUCTION

1.1. A question of Serre. Let S be a smooth cubic surface over a finite field \mathbb{F}_q . It is well-known (see for example [25, Thm. 27.1]) that

$$\#S(\mathbb{F}_q) = 1 + a(S)q + q^2$$

where $a(S)$ is the trace of the Frobenius element $\text{Fr}_q \in \text{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$ acting on the Picard group $\text{Pic } \bar{S} \cong \mathbb{Z}^7$ of \bar{S} .

Under a choice of isomorphism $\text{Aut}(\text{Pic } \bar{S}) \cong W(\mathbf{E}_6)$ with the Weyl group of the \mathbf{E}_6 -root system, the action of Fr_q gives rise to a conjugacy class $C(S)$ of $W(\mathbf{E}_6)$. An inspection of the character table of $W(\mathbf{E}_6)$ reveals that we have $a(S) \in \{-2, -1, 0, 1, 2, 3, 4, 5, 7\}$ (see [29, §2.3.3]). Serre has asked (*loc. cit.*) which values of the trace can actually arise for a given q . Swinnerton-Dyer [33] treated some special cases of this problem, and showed that the trace values $-2, 5$ arise for all q , whereas 7 occurs if and only if $q \neq 2, 3, 5$. Our first theorem extends these results and gives a complete answer to Serre's question.

Theorem 1.1. *Let q be a prime power.*

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- (1) For $-2 \leq a \leq 5$ and for all q , there exists a smooth cubic surface S over \mathbb{F}_q with $a(S) = a$.
- (2) There exists a smooth cubic surface S over \mathbb{F}_q with $a(S) = 7$ if and only if $q \neq 2, 3, 5$.

Note that a smooth cubic surface S with $a(S) = 7$ is *split*, i.e. all its lines are defined over \mathbb{F}_q . It has been known for a long time, prior to the work of Swinnerton-Dyer [33], that such a surface exists over \mathbb{F}_q if and only if $q \neq 2, 3, 5$; this result appears to be first due to Hirschfeld [14, Thm. 20.1.7].

We briefly explain the proof of Theorem 1.1. Recall that any smooth cubic surface over an *algebraically closed field* is the blow-up of \mathbb{P}^2 in 6 rational points in general position. Whilst this does not hold over other fields in general, it turns out that the trace values $a \in \{1, 2, 3, 4, 5, 7\}$ arise from cubic surfaces which are blow-ups of \mathbb{P}^2 in certain collections of *closed* points in general position. We therefore show that such collections exist over every finite field \mathbb{F}_q , via combinatorial arguments. The case $a = 0$ can be obtained by blowing-up certain collections of closed points of total degree 7, and contracting a line. The case $a = -1$ can be deduced from work of Rybakov [26], and of course the remaining case $a = -2$ was handled by Swinnerton-Dyer [33].

1.2. Corrections to Manin's and Swinnerton-Dyer's tables. Let S be a smooth cubic surface over a finite field \mathbb{F}_q . Building on work of Frame [11] and Swinnerton-Dyer [32], Manin constructed a table (Table 1 of [25, p. 176]) of the conjugacy classes of $W(\mathbf{E}_6)$ and their properties, such as the trace $a(S)$.

Urabe [34, 35] first noticed that Manin's table contains mistakes in Column 8, regarding the calculation of $H^1(\mathbb{F}_q, \text{Pic } \bar{S})$ (the issue being that $H^1(\mathbb{F}_q, \text{Pic } \bar{S})$ must have *square* order). In our investigation we found some new mistakes in Manin's table. These concern the Galois orbit on the lines, where the mistake can be traced back to Swinnerton-Dyer [32], and the *index* [25, §28.2] of the surface. The index is the size of the largest Galois invariant collection of pairwise skew lines over $\bar{\mathbb{F}}_q$.

Manin's table has a surface of index 2, which led him to state [25, Thm. 28.5(i)] that the index can only take one of the values 0, 1, 2, 3, 6. Our investigations reveal, however, that index 2 does not occur, and that index 5 can occur, hence [25, Thm. 28.5(i)] is false. The correct statement is the following.

Theorem 1.2. *Let S be a smooth cubic surface over a finite field. Then the index of S can only take one of the values 0, 1, 3, 5, 6.*

We constructed a corrected version of this table using **Magma** [2], which can be found in Section 7. For completeness, we also give geometric proofs of our corrected values.

Manin's book initiated a wave of interest in the arithmetic of del Pezzo surfaces, which continues to this day. The authors hope that it will be a useful addition to the literature to include a fully corrected version of Manin's table over 40 years after it was originally published.

1.3. Del Pezzo Surfaces. We also consider the analogue of Serre's question for other del Pezzo surfaces S , focusing on the interesting cases of degree $d \leq 4$.

One has [25, Thm. 23.9] a similar isomorphism $\text{Aut}(\text{Pic } \bar{S}) \cong W(\mathbf{E}_{9-d})$ (we follow Dolgachev's convention [7, §8.2.3], and define the \mathbf{E}_r -root system for any $3 \leq r \leq 8$). The trace of Frobenius $a(S)$ satisfies $a(S) \in \mathcal{A}_d$, where

$$\begin{aligned}\mathcal{A}_4 &= \{-2, -1, 0, 1, 2, 3, 4, 6\}, \\ \mathcal{A}_2 &= \{-6, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 8\}, \\ \mathcal{A}_1 &= \{-7, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 9\}.\end{aligned}\tag{1.1}$$

The set \mathcal{A}_4 can be deduced from Table 7.1 as follows: the blow-up of a quartic del Pezzo surface S in a rational point not on line is a cubic surface S' with a line, and one has $a(S') = a(S) + 1$ by Lemma 2.2. The values for \mathcal{A}_2 and \mathcal{A}_1 can be found in Urabe's tables [34] (the trace a is the exponent 1^a of 1 of the Frame symbol in *loc. cit.*). We give a complete classification of which traces can arise.

Theorem 1.3. *Let q be a prime power.*

- (1) *For $-2 \leq a \leq 4$ and for all q , there exists a quartic del Pezzo surface S over \mathbb{F}_q with $a(S) = a$.*
- (2) *There exists a quartic del Pezzo surface S over \mathbb{F}_q with $a(S) = 6$ if and only if $q \neq 2, 3$.*

Theorem 1.3 follows fairly readily from the method used to handle cubic surfaces. The following results require more work.

Theorem 1.4. *For each $a \in \mathcal{A}_2$ let B_a denote the set of prime powers q for which there does not exist a del Pezzo surface S of degree 2 over \mathbb{F}_q with $a(S) = a$. Then*

- (1) $B_a = B_{2-a}$;
- (2) $B_a = \emptyset$ for $a = 1, 2, 3$;
- (3) $B_a = \{2\}$ for $a = 4, 5$;
- (4) $B_6 = \{2, 3, 4\}$;
- (5) $B_8 = \{2, 3, 4, 5, 7, 8\}$.

Theorem 1.5. *For each $a \in \mathcal{A}_1$ let B_a denote the set of prime powers q for which there does not exist a del Pezzo surface S of degree 1 over \mathbb{F}_q with $a(S) = a$. Then*

- (1) $B_a = B_{2-a}$;
- (2) $B_a = \emptyset$ for $a = 1, 2, 3, 4$;
- (3) $B_5 = \{2\}$;
- (4) $B_6 = \{2, 3, 4, 5\}$;
- (5) $B_7 = \{2, 3, 4, 5, 7, 8, 9\}$;
- (6) $B_9 = \{2, 3, 4, 5, 7, 8, 9, 11, 13, 17\}$.

The cases of trace at least 1 are handled similarly to the proof of Theorem 1.1, by considering various configurations of closed points in general position in \mathbb{P}^2 (at least this works for large q). The arguments here are more delicate than the case of cubic surfaces and more difficulties arise. For small q we also need to consider some surfaces which are not of maximal index, and show either their existence or non-existence. We do this using a range of combinatorial and geometric techniques, such as the theory of conic bundles and appealing to the classification of possible Galois actions by Urabe [34]. **Magma** is employed to help with a few remaining difficult cases.

To handle trace less than 1, we use an amusing trick: any del Pezzo surface S of degree 2 or 1 admits a automorphism of order 2, hence a non-trivial quadratic twist over \mathbb{F}_q , which we denote by S_σ . A counting argument shows that $a(S) + a(S_\sigma) = 2$, in particular, on performing a quadratic twist we handle the remaining trace values (this explains the above symmetry $B_a = B_{2-a}$).

Note that a del Pezzo surface S over \mathbb{F}_q of degree $d \leq 7$ has trace $10 - d$ if and only if S is the blow-up of \mathbb{P}^2 in $9 - d$ rational points in general position. In particular, for $r \leq 8$, our results give a complete classification of those q for which there exist r rational points in \mathbb{P}^2 over \mathbb{F}_q in general position. The new cases are as follows.

Corollary 1.6.

- (1) *There exist 7 rational points in general position in \mathbb{P}^2 over \mathbb{F}_q if and only if $q \geq 9$.*
- (2) *There exist 8 rational points in general position in \mathbb{P}^2 over \mathbb{F}_q if and only if $q = 16$ or $q \geq 19$.*

Knowledge about the possible number of rational points on del Pezzo surfaces over finite fields is often required in proofs, especially over small finite fields (see e.g. the proof of [27, Thm. 1]). The authors hope that our results will find use in such proofs, and will, for example, assist with unirationality considerations for del Pezzo surfaces of degree 1 over finite fields.

1.4. An inverse Galois problem. Generalising Serre's question, one may ask which conjugacy classes of $W(\mathbf{E}_{9-d})$ arise for some del Pezzo surface of degree d over \mathbb{F}_q . We are able to answer this question for sufficiently large q . For uniformity of exposition we focus on the more interesting case $d \leq 6$.

For a field k , we denote by $\mathcal{S}_d(k)$ the set of isomorphism classes of del Pezzo surfaces of degree d over k . For a del Pezzo surface S of degree d over a finite field, we denote by $C(S)$ the conjugacy class of $W(\mathbf{E}_{9-d})$ obtained from the action of Fr_q on $\text{Pic } \bar{S}$. We want to study the distribution of $C(S)$ as q grows. As is common when counting objects up to isomorphism, we weight each surface by the size of its automorphism group. For a conjugacy class C of $W(\mathbf{E}_d)$ we define

$$\tau_d(C) = \lim_{q \rightarrow \infty} \frac{\sum_{S \in \mathcal{S}_d(\mathbb{F}_q), C(S)=C} \frac{1}{|\text{Aut } S|}}{\sum_{S \in \mathcal{S}_d(\mathbb{F}_q)} \frac{1}{|\text{Aut } S|}}. \quad (1.2)$$

Theorem 1.7. *The limit (1.2) exists and we have*

$$\tau_d(C) = \frac{\#C}{\#W(\mathbf{E}_{9-d})}.$$

Values for $\#C/\#W(\mathbf{E}_6)$ when $d = 3, 2, 1$ can be found in Table 7.1, and [34, Tab. 1, Tab. 2], respectively. Theorem 1.7 follows without too much difficulty from known results in the literature, and is proved using a suitable version of the Chebotarev density theorem; we use the version given by Ekedahl in [9], which is proved using Deligne's "Weil II paper" [5].

From this we immediately obtain the following.

Corollary 1.8. *For $q \gg 1$, the inverse Galois problem for del Pezzo surfaces of degree d over \mathbb{F}_q is solvable.*

Of course Corollary 1.8 is only new for small d . For $d \geq 5$ it is known that every conjugacy class is realisable over every finite field (see e.g. [31, Thm. 3.1.3] for $d = 5$ and [1, Thm. 3.5], [3, Thm. 4.2] for $d = 6$, respectively). For $d \leq 4$, however, Corollary 1.8 appears to be new.

The proof of Corollary 1.8 could in theory be made effective. In the case $d = 3$, for example, one would need upper bounds for the dimensions of cohomology groups with compact support for certain ℓ -adic sheaves on the open subset $U \subset \mathbb{P}^{19}$ that parametrises smooth cubic surfaces in \mathbb{P}^3 . It seems doubtful however that such bounds would be small enough for the remaining cases to be amenable to machine computation for small d . It would be interesting to obtain a complete classification for all q and d . Note that of course the conclusion of Corollary 1.8 does not hold for all q , as Theorem 1.1 illustrates.

Automorphism groups of generic del Pezzo surface of degree d are well-known. As a consequence, we are able to count without weighting by the automorphism group in Theorem 1.7 whenever $d \leq 3$, to obtain the following.

Theorem 1.9. *For $d \leq 3$ we have*

$$\lim_{q \rightarrow \infty} \frac{\#\{S \in \mathcal{S}_d(\mathbb{F}_q) : C(S) = C\}}{\#\mathcal{S}_d(\mathbb{F}_q)} = \frac{\#C}{\#W(\mathbf{E}_{9-d})}.$$

1.4.1. *Vertical Sato-Tate.* As an application of Theorem 1.9, we address the more refined question of the distribution of the trace values for del Pezzo surfaces over finite fields \mathbb{F}_q , as $q \rightarrow \infty$. For $a \in \mathbb{Z}$ we let

$$\tau_d(a) = \lim_{q \rightarrow \infty} \frac{\#\{S \in \mathcal{S}_d(\mathbb{F}_q) : a(S) = a\}}{\#\mathcal{S}_d(\mathbb{F}_q)}. \quad (1.3)$$

Theorem 1.9 and an enumeration of the conjugacy classes of $W(\mathbf{E}_{9-d})$, gives the following.

Corollary 1.10. *For $d \leq 3$, the limit (1.3) exists and takes the following values:*

a	$\tau_3(a)$	$\tau_2(a)$	$\tau_1(a)$
−7			1/696729600
−6		1/2903040	
−5			1/5806080
−4		1/46080	1/311040
−3		1/4320	653/4976640
−2	1/648	13/3072	2267/518400
−1	77/1152	169/3240	225157/4147200
0	9/40	34423/138240	262679/1088640
1	347/864	653/1680	442169/1105920
2	91/360	34423/138240	262679/1088640
3	3/64	169/3240	225157/4147200
4	1/216	13/3072	2267/518400
5	1/1440	1/4320	653/4976640
6		1/46080	1/311040
7	1/51840		1/5806080
8		1/2903040	
9			1/696729600

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2. GENERALITIES

2.1. The a -invariant.

Definition 2.1. Let S be a smooth geometrically rational projective surface over a finite field \mathbb{F}_q . We define $a(S)$ to be the trace of the Frobenius element of $\text{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$ acting on $\text{Pic } \bar{S}$.

By [25, Thm. 27.1] we have

$$\#S(\mathbb{F}_q) = 1 + a(S)q + q^2. \quad (2.1)$$

In particular, the invariant $a(S)$ plays a similar rôle to the a -invariant of an elliptic curve.

Lemma 2.2. *Let S be a smooth geometrically rational projective surface over a finite field \mathbb{F}_q . Let $x \in S$ be a closed point and $\text{Bl}_x S$ be the blow-up of S at x . Then*

$$a(\text{Bl}_x S) = \begin{cases} a(S) + 1, & x \in S(\mathbb{F}_q), \\ a(S), & x \notin S(\mathbb{F}_q). \end{cases}$$

Proof. If $x \in S(\mathbb{F}_q)$, then blowing-up replaces a rational point by a copy of \mathbb{P}^1 , which has $1 + q$ rational points. If $x \notin S(\mathbb{F}_q)$, then blowing-up neither removes nor adds any rational points. \square

2.2. Del Pezzo surfaces. We recall some facts about del Pezzo surfaces, which can be found in [25] and [7, §8]. A del Pezzo surface S over a field k is a smooth projective surface over k with ample anticanonical bundle. We define the degree d of S to be $(-K_S)^2$; we have $1 \leq d \leq 9$. A *line* on S is a smooth rational curve $L \subset S$ with $L^2 = -1$. For $7 \leq d \leq 3$, such a curve is a line in the usual sense with respect to the anticanonical embedding. We say that S is *split* if the natural map $\text{Pic } S \rightarrow \text{Pic } \bar{S}$ is an isomorphism; for $d \leq 7$ this is equivalent to all the lines of S over \bar{k} being actually defined over k .

Definition 2.3. Let $r \leq 8$ and let k be a field. We say that a collection of distinct points $P_1, \dots, P_r \in \mathbb{P}^2(k)$ lie in general position if the following hold.

- (1) No 3 are collinear.
- (2) No 6 lie on a conic.
- (3) No 8 lie on a cubic with a singularity at one of the points.

We say that a collection of distinct *closed* points of \mathbb{P}^2 of total degree less than 8 lie in general position if the corresponding points over \bar{k} lie in general position.

When k is perfect, we will often abuse notation and identify a closed point P with the Galois invariant collection $P_{\bar{k}}$ of rational points over \bar{k} .

As proved by Manin [25, Thm. 24.5] for $r \leq 6$ and D  mazure [6, Thm. 1] for $r = 7, 8$, a collection of $r \leq 8$ rational points lie in general position if and only if their blow-up is a del Pezzo surface S . For $r \leq 7$, the lines on S consist of the exceptional curves of the blow-ups of the points, together with the strict transforms of the following curves [25, Thm. 26.2]:

- (1) Lines through two of the points.
- (2) Conics through five of the points.
- (3) Cubic curves through seven of the points, with a double point at exactly one of them.

There is a simple criterion to check whether 5 points lie in general position.

Lemma 2.4. *Let $P_1, \dots, P_5 \in \mathbb{P}^2(k)$ be five distinct points. Then P_1, \dots, P_5 lie in general position if and only if they lie on a smooth conic.*

Proof. Clearly P_1, \dots, P_5 lie on some conic C . If C is singular, then it is either a union of two lines or a double line; in particular 3 of the points are collinear. If C is smooth, then any line intersects C in at most 2 points, hence no 3 are collinear. \square

Our next lemma is a linear algebra criterion to check whether a collection of points lies in general position, which will be used for computations. It makes explicit the well-known fact that the collection of all $r \leq 8$ rational points of \mathbb{P}^2 not in general position, viewed as a subset of $(\mathbb{P}^2)^r$, is not Zariski dense.

Lemma 2.5. *Let k be a field and let $P_i = (x_i : y_i : z_i) \in \mathbb{P}^2(k)$ be a collection of distinct rational points, for $1 \leq i \leq r$.*

- (1) *If $r = 3$, then P_1, P_2, P_3 are collinear if and only if*

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0. \quad (2.2)$$

- (2) *If $r = 6$, then P_1, \dots, P_6 lie on a conic if and only if the matrix $M \in M_{6,6}(k)$ whose i th row is*

$$(x_i^2 \ y_i^2 \ z_i^2 \ x_i y_i \ x_i z_i \ y_i z_i) \quad (2.3)$$

has determinant 0.

- (3) *If $r = 8$, for each $1 \leq i \leq 8$ consider the matrix $M_i \in M_{11,10}(k)$ whose j th row for $1 \leq j \leq 8$ is*

$$(x_j^3 \ y_j^3 \ z_j^3 \ x_j^2 y_j \ x_j^2 z_j \ x_j y_j^2 \ y_j^2 z_j \ x_j z_j^2 \ y_j z_j^2 \ x_j y_j z_j),$$

and whose last three rows are

$$\begin{pmatrix} 3x_i^2 & 0 & 0 & 2x_i y_i & 2x_i z_i & y_i^2 & 0 & z_i^2 & 0 & y_i z_i \\ 0 & 3y_i^2 & 0 & x_i^2 & 0 & 2x_i y_i & 2y_i z_i & 0 & z_i^2 & x_i z_i \\ 0 & 0 & 3z_i^2 & 0 & x_i^2 & 0 & y_i^2 & 2x_i z_i & 2y_i z_i & x_i y_i \end{pmatrix}.$$

Then P_1, \dots, P_8 lie on a cubic with a singularity at P_i if and only if M_i has a non-trivial kernel.

Proof. The first part is elementary. For the second, consider a conic of the shape

$$C : b_0X^2 + b_1Y^2 + b_2Z^2 + b_3XY + b_4XZ + b_5YZ = 0 \quad \subset \mathbb{P}^2$$

where $b_i \in k$. The condition that $P_i \in C$ imposes linear conditions on the b_i . The resulting system of linear equations has a non-trivial solution if and only if the determinant of (2.3) is zero. For cubics, a vector that lies in the kernel of the first 8 rows of the matrix M_i corresponds to a cubic curve which contains the given 8 points. The last three rows determine whether the partial derivatives of this cubic vanish at the point P_i , whence the result. \square

2.3. Conic bundles. In this paper, a conic bundle over a field k is a smooth projective surface S over k together with a morphism $\pi : S \rightarrow \mathbb{P}^1$ all of whose fibres are isomorphic to conics. We say that $\pi : S \rightarrow \mathbb{P}^1$ is *relatively minimal* if the fibre over each point is irreducible.

Lemma 2.6. *Let $\pi : S \rightarrow \mathbb{P}^1$ be a relatively minimal conic bundle over \mathbb{F}_q . Then*

(1) *The set*

$$\{x \in \mathbb{P}^1 : \pi^{-1}(x) \text{ is singular}\}$$

has even cardinality.

(2) *We have*

$$a(S) = 2 - \#\{x \in \mathbb{P}^1(\mathbb{F}_q) : \pi^{-1}(x) \text{ is singular}\}.$$

Proof. Part (1) follows from the fundamental exact sequence from class field theory; see [26, Cor. 2.10]. Part (2) follows from (2.1) and an elementary count: as S is smooth and π relatively minimal, the fibre over a rational point is either a smooth plane conic, hence contains $1 + q$ points, or is singular and contains exactly 1 rational point (being isomorphic to 2 lines meeting in a single point after a quadratic extension). \square

3. CUBIC SURFACES AND QUARTIC DEL PEZZO SURFACES

In this section we prove Theorem 1.1, following the strategy outlined in the introduction. For trace at least 1, it suffices to consider the existence of the following collections of closed points in general position.

a	Class	Points in general position to blow-up
7	C_1	6 rational points
5	C_{16}	4 rational points, one closed point of degree 2
4	C_6	3 rational points, one closed point of degree 3
3	C_{18}	2 rational points, one closed point of degree 4
2	C_{15}	1 rational points, one closed point of degree 5
1	C_{23}	One closed point of degree 6

The reader may verify with Lemma 2.2 that the given blow-ups yield the claimed trace a . We have also included the corresponding conjugacy class, as can be found in Table 7.1. The case of trace 0 is handled using a similar method, and the remaining traces -1 and -2 can be handled using [26] and [33], respectively. We briefly explain at the end the proof of Theorem 1.3.

3.1. Proof of Theorem 1.1. Whilst the cases $a = 7, 5$ were already dealt with by Swinnerton-Dyer [33], we give alternative proofs in the spirit of our method. The lemmas proved here will also be required in the sequel.

$a = 7$. First note that smooth cubic surfaces of trace 7 are exactly those which are blow-ups of \mathbb{P}^2 in 6 rational points in general position. We start with some basic lemmas.

Lemma 3.1. *There exist five points $P_1, \dots, P_5 \in \mathbb{P}^2(\mathbb{F}_q)$ in general position if and only if $q \geq 4$.*

Proof. This follows from Lemma 2.4 and the fact that a smooth conic has $q + 1$ rational points over \mathbb{F}_q . \square

Suppose now that $q \geq 4$. Let $P_1, \dots, P_5 \in \mathbb{P}^2(\mathbb{F}_q)$ be in general position and let C be the smooth conic passing through them. We now compute the number of rational points on the configuration \mathfrak{C} of ten lines through these five rational points, together with the conic C .

Lemma 3.2. *For $q \geq 4$, we have $\#\mathfrak{C}(\mathbb{F}_q) = 11q - 24$.*

Proof. Let \mathfrak{L} denote the union of the ten lines determined by P_1, \dots, P_5 . Since the number of points on a line of \mathfrak{L} where four lines meet is 2 and the number of points of a line of \mathfrak{L} where only two lines meet is 3, the number of rational points of \mathfrak{L} is

$$\#\mathfrak{L}(\mathbb{F}_q) = 10(q + 1 - 5) + \frac{10 \cdot 2}{4} + \frac{10 \cdot 3}{2}.$$

Add to this the number of points on the conic C not on \mathfrak{L} (namely, $q - 4$), and the result follows. \square

Corollary 3.3. *There exist six points $P_1, \dots, P_6 \in \mathbb{P}^2(\mathbb{F}_q)$ in general position if and only if q is not 2, 3, or 5.*

Proof. By Lemma 3.2, we have $\#(\mathbb{P}^2(\mathbb{F}_q) \setminus \mathfrak{C}(\mathbb{F}_q)) = (q - 5)^2$. For $q \geq 4$, this is strictly positive if and only if $q \neq 5$. \square

This completes the case $a = 7$.

We now let q be an arbitrary prime power and choose a smooth conic C over \mathbb{F}_q .

$a = 5$. The conic C contains three rational points P_1, P_2, P_3 and a closed point of degree 2; these lie in general position by Lemma 2.4. Denote, by ℓ_1, ℓ_2 , and ℓ_3 the lines through each of the pairs of P_1, P_2, P_3 . Denote by P_4 and P_5 the points over \mathbb{F}_{q^2} determined by the closed point and ℓ the line through them. One easily sees that the line connecting P_i with P_j , for $i \in \{1, 2, 3\}$ and $j \in \{4, 5\}$ has only P_i as an \mathbb{F}_q -rational point. Let $L = \ell_1 \cup \ell_2 \cup \ell_3 \cup \ell$ and $X = L \cup C$. From the above considerations it follows that

$$\#X(\mathbb{F}_q) = \#L(\mathbb{F}_q) + q - 2 = 4(q + 1) - 6 + q - 2 = 5q - 4$$

There exists a sixth point P_6 such that P_1, \dots, P_6 are in general position if and only if $0 < q^2 + q + 1 - 5q + 4 = (q - 2)^2 + 1$, which is always satisfied.

$a = 4$. Choose a point $P_1 \in C(\mathbb{F}_{q^3}) \setminus C(\mathbb{F}_q)$. Denote by P_2 and P_3 the conjugates of P_1 and choose rational points P_4 and P_5 on C . By Lemma 2.4 these points lie in general position. Let ℓ be the line through P_4 and P_5 . There exists a sixth \mathbb{F}_q -rational point P_6 such that P_1, \dots, P_6 are in general position if and only if

$$0 < q^2 + q + 1 - \#(\ell \cup C)(\mathbb{F}_q) = (q - 1)^2 + q,$$

which is always true.

$a = 3$. Let $P_1 \in C(\mathbb{F}_{q^4}) \setminus C(\mathbb{F}_{q^2})$. Denote by P_2, P_3, P_4 the conjugates of P_1 and let $P_5 \in C(\mathbb{F}_q)$; these lie in general position by Lemma 2.4. One easily sees that the lines through two of the P_i contain only a single rational point not on C , which we denote by Q , given as the intersection of the line through P_1 and P_3 and the line through P_2 and P_4 . We may conclude as before by noting that $0 < q^2 + q + 1 - \#(C \cup Q)(\mathbb{F}_q) = q^2 - 1$.

$a = 2$. Choose a point in $C(\mathbb{F}_{q^5}) \setminus C(\mathbb{F}_q)$. The conjugates P_i of this point give a closed point P of degree 5 on C , which lies in general position by Lemma 2.4. One easily sees that lines through pairs of the P_i contain no \mathbb{F}_q -points. We conclude as before by noting that $q^2 + q + 1 - \#C(\mathbb{F}_q) > 0$.

$a = 1$. Let $\alpha_1, \dots, \alpha_6$ be a normal basis of \mathbb{F}_{q^6} over \mathbb{F}_q , i.e. $\alpha_{i+1} = \alpha_i^q$ for $i = 1, \dots, 6$, where the subscripts are taken modulo 6. Write $P_i = [1 : \alpha_i : \alpha_i^3]$ and note that the collection $\{P_1, \dots, P_6\}$ forms a closed point of degree 6. For P_i, P_j , and P_k , the determinant (2.2) here is

$$(\alpha_k - \alpha_i)(\alpha_k - \alpha_j)(\alpha_j - \alpha_i)(\alpha_i + \alpha_j + \alpha_k) \neq 0,$$

which ensures that P_i, P_j , and P_k are not collinear by Lemma 2.5. Similarly, the determinant of the matrix (2.3) is

$$\prod_{1 \leq i < j \leq 6} (\alpha_j - \alpha_i)(\alpha_1 + \dots + \alpha_6) \neq 0,$$

hence the points lie in general position by Lemma 2.5.

$a = 0$. We will show the existence of a closed point of degree 5 and a closed point of degree 2 in general position on \mathbb{P}^2 . Having done this, we blow up these points to obtain a del Pezzo surface S' of degree 2 with $a(S') = 1$. This surface contains a line, corresponding to the line in \mathbb{P}^2 passing through the closed point of degree 2. By Lemma 2.2, the blow-down of this line yields a cubic surface S with $a(S) = 0$, which one checks has class C_{25} .

Choose a closed point of degree 5 on C ; this lies in general position by Lemma 2.4. As in the case $a = 2$, none of the lines through pairs of the 5 conjugate points contains an \mathbb{F}_q -point. As $(\#\mathbb{P}^2(\mathbb{F}_{q^2}) - \#\mathbb{P}^2(\mathbb{F}_q) - \#C(\mathbb{F}_{q^2})) = q^4 - q^2 - q - 1 > 0$, we see that there is a closed point of degree 2 in \mathbb{P}^2 which does not lie on C . Moreover, a simple consideration of the Galois action shows that no 3 of these points are collinear.

It needs only be checked that no 6 of these points lie on a conic. By construction, the closed point of degree 2 does not lie on C . So suppose that there is a conic C' passing through the closed point of degree 2 and four of the quintic points (underlying the closed point of degree 5); C' is defined over \mathbb{F}_{q^5} . Let Fr_q be

a generator of $\text{Gal}(\mathbb{F}_{q^5}/\mathbb{F}_q)$, and consider C'^{Fr_q} . On the one hand, C' and C'^{Fr_q} are different conics; but on the other hand they have five points in common. This is a contradiction.

$a = -1$. Such a surface was constructed by Rybakov in the proof of [26, Thm. 3.2]. We briefly recall the details to clarify why the construction works for all q (this will construct the class C_{19}).

Let $x_1, x_2, x_3 \in \mathbb{P}^1(\mathbb{F}_q)$ be distinct and take $x_4 \in \mathbb{P}^1$ a closed point of degree 2; these exist for any q . Let $\pi : S \rightarrow \mathbb{P}^1$ be a relatively minimal conic bundle over \mathbb{P}^1 whose singular fibres lie exactly above x_1, x_2, x_3, x_4 ; such a surface exists by class field theory (see the proof of [26, Thm. 2.11]). By [15, Thm. 5] we see that S is a cubic surface, and Lemma 2.6 implies that $a(S) = -1$, as required.

$a = -2$. This case was already handled by Swinnerton-Dyer [33].

This completes the proof of Theorem 1.1. \square

3.2. Proof of Theorem 1.3. The result for $a = 6$ follows from Lemma 3.1. For $a \geq 1$, it suffices to note that in the proof of Theorem 1.1 we showed the existence of a smooth cubic surface S over \mathbb{F}_q with $a(S) = a + 1$ and such that S contains a line; contracting this line yields the required quartic del Pezzo surface by Lemma 2.2. For $a = -2, 0$, the required surfaces have been constructed by Rybakov [26, Thm. 3.2] (these have decompositions X and $XVIII$, in the notation of Table 7.1, respectively).

This leaves only the value $a = -1$. This is handled in a similar manner to the $a = 0$ case of cubic surfaces. Namely, one shows that the existence of two closed points of degree 2 and a closed point of degree 3 in general position for any q . The blow-up of these is a del Pezzo surface S' of degree 2 with two skew lines. The blow-down of these lines gives the required quartic del Pezzo surface, by Lemma 2.2. \square

4. DEL PEZZO SURFACES OF DEGREE 2

In this section we establish Theorem 1.4.

4.1. Definitions and basic properties. Let k be a field. Any del Pezzo surface S of degree 2 over k can be written in the form

$$w^2 + f_2(x, y, z)w = f_4(x, y, z) \quad \subset \mathbb{P}(1, 1, 1, 2) \quad (4.1)$$

where $\deg f_i = i$.

4.1.1. The ramification curve. The anticanonical map $\pi : S \rightarrow \mathbb{P}^2$ is given by the projection $(x : y : z : w) \mapsto (x : y : z)$, and realises S as a double cover of \mathbb{P}^2 . The behaviour in characteristic 2 is slightly different; a good general reference for double covers in characteristic 2 is [4, §0.1]. The morphism π is separable in all characteristics.

When $\text{char } k \neq 2$ the double cover is ramified over the smooth quartic curve $B : f_4(x, y, z) = 0$. We define the ramification curve to be $R = \pi^{-1}(B)_{\text{red}}$, i.e the reduced subscheme underlying $\pi^{-1}(B)$. When $\text{char } k = 2$ the branch curve B is the plane conic $f_2(x, y, z) = 0$ (this can be reducible or non-reduced). Following

the convention of [4, p. 14], in this case we define the ramification curve to be $R = \pi^{-1}(B)$ (this can be reduced or non-reduced; see the proof of Lemma 4.1).

We have the following lemma on the geometry of the ramification curve, which will be required for the proof of Theorem 1.5. We define the genus $g(C)$ of an irreducible (possibly singular or non-reduced) projective curve C to be the genus of the normalisation of C_{red} .

Lemma 4.1. *Let S be a del Pezzo surface of degree 2 over an algebraically closed field k with ramification curve R .*

- *If $\text{char } k \neq 2$ then R is irreducible smooth and of genus 3.*
- *If $\text{char } k = 2$ then R has at most 2 irreducible components, and each irreducible component has genus 0.*

Proof. When $\text{char } k \neq 2$ the result is clear, as $R \cong B$ is a smooth plane quartic. So assume that $\text{char } k = 2$. Here R has the equation

$$R: \quad f_2(x, y, z) = 0, \quad w^2 = f_4(x, y, z) \quad \subset S.$$

We consider the various possibilities for B .

- (1) B is a smooth plane conic. Here R is irreducible and reduced, but may be singular. The morphism $R \rightarrow B$ is purely inseparable of degree 2. Let $N \rightarrow R$ be the normalisation of R . The induced map $N \rightarrow B$ is still purely inseparable of degree 2, hence $g(N) = g(B)$ (see e.g. [18, Lem. 8.6.6]) and thus $g(R) = 0$.
- (2) $B = L_1 \cup L_2$ is a union of 2 distinct lines. Each $R_i := \pi^{-1}(L_i)$ is irreducible and reduced and the map $R_i \rightarrow L_i$ is purely inseparable of degree 2. As in the previous case, we find that $g(R_i) = 0$.
- (3) $B = L^2$ is a double line. Here R is non-reduced, but $R_{red} \rightarrow L$ is purely inseparable of degree 2. As above, we conclude that $g(R) = 0$.

This completes the proof. \square

4.1.2. Quadratic twists. The double cover π induces an involution of S , called the *Geiser involution*. We may therefore twist by some $\alpha \in H^1(k, \mathbb{Z}/2\mathbb{Z})$, to obtain the *quadratic twist* S_α of S by α . If $k = \mathbb{F}_q$ is a finite field, then S admits a unique non-trivial quadratic twist up to isomorphism; we denote the choice of such a twist by S_σ .

For completeness we give explicit equations for the quadratic twists, though these will not be required in the sequel. When $\text{char } k \neq 2$, we may choose the equation (4.1) so that $f_2 = 0$. Kummer theory gives $H^1(k, \mathbb{Z}/2\mathbb{Z}) = k^*/k^{*2}$, and for $\alpha \in k^*$ the quadratic twist S_α has the form

$$\alpha w^2 = f_4(x, y, z).$$

When $\text{char } k = 2$, as π is separable, one may write down equations for S_α using Artin-Schreier theory instead of Kummer theory. Artin-Schreier theory gives $H^1(k, \mathbb{Z}/2\mathbb{Z}) \cong k/\wp k$, where $\wp(\alpha) = \alpha^2 - \alpha$. For $\alpha \in k$, the associated quadratic twist S_α is given by

$$w^2 + f_2(x, y, z)w = f_4(x, y, z) + \alpha f_2(x, y, z)^2.$$

Lemma 4.2. *Let S be a del Pezzo surface of degree 2 over a finite field \mathbb{F}_q and S_σ its non-trivial quadratic twist. Then*

$$a(S) + a(S_\sigma) = 2.$$

Proof. Let $\pi : S \rightarrow \mathbb{P}^2$ (resp. $\pi_\sigma : S_\sigma \rightarrow \mathbb{P}^2$) be the associated double cover of \mathbb{P}^2 . Let $x \in \mathbb{P}^2(\mathbb{F}_q)$. If x lies in the branch locus, then $\pi^{-1}(x)$ and $\pi_\sigma^{-1}(x)$ both have a single rational point (if $\text{char } k \neq 2$ this is clear; if $\text{char } k = 2$ then one observes, as in the proof of Lemma 4.1, that the map $R_{\text{red}} \rightarrow B_{\text{red}}$ is purely inseparable). If x does not lie in the branch locus, then one of $\pi^{-1}(x)$ or $\pi_\sigma^{-1}(x)$ contains exactly two rational points, and the other none. Taking these contributions together we obtain

$$\#S(\mathbb{F}_q) + \#S_\sigma(\mathbb{F}_q) = 2\#\mathbb{P}^2(\mathbb{F}_q) = 2(q^2 + q + 1).$$

The result follows on recalling (2.1). \square

Remark 4.3. Analogues of Lemma 4.2 for elliptic curves are well-known; see for example Exercises 61 and 62 of [8].

4.2. Proof of Theorem 1.4. Lemma 4.2 implies that it suffices to consider the case of trace at least 1. We shall handle these cases using a similar strategy to the proof of Theorem 1.1.

$a = 8$. Here we are concerned with characterizing those q for which there exist 7 rational points in $\mathbb{P}^2(\mathbb{F}_q)$ in general position. There are already some partial results in the literature concerning this problem (see e.g. [17, Lem. 68] and the proof of [19, Lem. 2.1]). These authors however assume throughout that q is *odd*, as further difficulties arise in the case of even q . We give new proofs which apply uniformly to all q .

Lemma 4.4. *For $q \geq 8$, there exist six distinct points in general position in $\mathbb{P}^2(\mathbb{F}_q)$ such that, if \mathfrak{C} denotes the configuration of the fifteen lines through pairs of these points, together with the six conics through any five of them, we have $\#\mathfrak{C}(\mathbb{F}_q) \leq 21q - 64$.*

Proof. Let P_1, \dots, P_6 be any six rational points in general position and let \mathfrak{L} denote the union of the fifteen lines through pairs of them. Let n_2 (resp. n_3) be the number of points where exactly two (resp. three) of these lines meet. One easily sees that $n_2 + 3n_3 = 45$, and we have

$$\#\mathfrak{L}(\mathbb{F}_q) = 15(q + 1) - 4 \cdot 6 - 2n_3 - n_2 = 15q - 54 + n_3.$$

Let C_i denote the smooth conic that passes through the points P_1, \dots, P_6 except P_i and let t_i be the number of points P_j , with $j \neq i$, such that the line connecting P_i and P_j is tangent to C_i . The conic C_i adds $q + 1 - 10 + t_i$ points to the configuration. Since the conics C_i have no points in common outside the P_i , we obtain

$$\#\mathfrak{C}(\mathbb{F}_q) = 21q - 108 + n_3 + \sum_{i=1}^6 t_i.$$

The trivial bounds $n_3 \leq 15$ and $\sum_{i=1}^6 t_i \leq 30$ yield $\#\mathfrak{C}(\mathbb{F}_q) \leq 21q - 63$. The lemma will follow from the fact that there is a choice of P_1, \dots, P_6 for which $\sum_{i=1}^6 t_i \leq 29$. Indeed, let P_1, \dots, P_5 be any five points in general position. Let

ℓ be a line through P_5 of slope distinct from the slopes of the lines connecting P_5 with P_1, P_2, P_3 , and P_4 ; and distinct from the slope of the tangent line at P_5 to the conic C through P_1, \dots, P_5 . By construction, ℓ intersects C at P_5 and at a second point, say Q . Let P_6 be a point on ℓ distinct from P_5, Q , and the intersection of ℓ with any of the lines through a pair of points of P_1, \dots, P_4 . We can do this as long as $q + 1 > 2 + 6$, which holds by hypothesis. \square

Proposition 4.5. *Seven points $P_1, \dots, P_7 \in \mathbb{P}^2(\mathbb{F}_q)$ in general position exist if and only if $q \geq 9$.*

Proof. Let S be the del Pezzo surface of degree 2 obtained by blowing-up these points. The blow-down of a line L is a split smooth cubic surface S' , and the image of L is a rational point not on a line. However, as proved by Hirschfeld in [14, Thm. 20.3.9, Thm. 20.3.10], there is no such smooth cubic surface when $q \leq 8$.

Suppose now that $q \geq 16$. By Lemma 4.4, there exist six points P_1, \dots, P_6 in general position for which $\#\mathfrak{C}(\mathbb{F}_q) \leq 21q - 64$. Since

$$\#\mathbb{P}^2(\mathbb{F}_q) \setminus \mathfrak{C} \geq (q - 10)^2 - 35 > 0,$$

we can choose P_7 to be any point in the non-empty complement of \mathfrak{C} in $\mathbb{P}^2(\mathbb{F}_q)$. For the remaining values $q = 9, 11, 13$, rational points in general position can be easily found using a computer and Lemma 2.5 (alternatively, as these q are *odd*, the necessary existence follows from [17, Lem. 68]). \square

$a = 6$. According to [34, Tab. 1], a del Pezzo surface S of degree 2 with $a(S) = 6$ must be the blow-up of \mathbb{P}^2 in 5 rational points and a closed point of degree 2 lying in general position.

Note that after a quadratic extension we would obtain 7 rational points in \mathbb{P}^2 over \mathbb{F}_{q^2} . By Proposition 4.5, this is not possible for $q^2 \leq 8$, i.e. for $q = 2$. The values $q = 3, 4$ are small enough that one can enumerate the relevant collections of points in \mathbb{P}^2 on a computer, and verify their non-existence using Lemma 2.5.

Now consider $q \geq 5$. Let C be a smooth conic over \mathbb{F}_q , and let P denote a point on C defined over \mathbb{F}_{q^2} but not over \mathbb{F}_q ; let P^σ denote its Galois conjugate, which is also on C . Choose three \mathbb{F}_q -rational points P_1, P_2, P_3 on the conic; then the five points $P_1, P_2, P_3, P, P^\sigma$ are in general position by Lemma 2.4.

Consider the configuration \mathfrak{C} of the 10 lines through pairs of these five points, as well as the conic C we started with. Again using the argument as in the $a = 5$ case of cubic surfaces, we see that this configuration contains exactly $5q - 4$ rational points. Let Q be a rational point *not* on this configuration; then the six points $P_1, P_2, P_3, P, P^\sigma, Q$ are in general position.

We extend the configuration \mathfrak{C} to include the five lines from Q to the other five points, as well as the 5 conics defined by Q and four of the other five points (in the sequel we will refer to these as the *five new lines* and *five new conics* respectively, and rational points on them *not* on \mathfrak{C} as *new rational points*). It is not possible to compute exactly how many rational points are contained in this extended configuration (due to the possible existence of Eckhardt points); though it is not too difficult to obtain a reasonable upper bound. Only three of the five new lines are rational, and the maximum number of new rational points on the five new lines is $3q - 5$ (this comes from minimising the intersections of

the new lines with \mathfrak{C}). Similarly, the maximum number of new rational points on the five new conics is $3q - 6$. Thus, the maximum possible number of rational points on the extended configuration is $11q - 15$; comparing this with the total number $q^2 + q + 1$ of rational points on the plane, we see that we can choose a rational seventh point lying in general position relative to the first six provided $q \geq 9$.

For the remaining values $q = 5, 7, 8$, a computer search reveals that the required configurations exist.

$a = 5$. According to [34, Tab. 1], a del Pezzo surface S of degree 2 with $a(S) = 5$ must be the blow-up of \mathbb{P}^2 in 4 rational points and a closed point of degree 3 in general position. As in the case $a = 6$, we may pass to a cubic extension and apply Proposition 4.5 to see that such a configuration does not exist when $q = 2$.

Let $q \geq 3$ and let C be a smooth conic over \mathbb{F}_q . Let $P \in C(\mathbb{F}_{q^3}) \setminus C(\mathbb{F}_q)$. The conjugates of P yield a closed point of degree 3. Choose two rational points on C . These 5 points are in general position by Lemma 2.4.

We next choose a sixth \mathbb{F}_q -rational point which lies in general position relative to these five. Obviously we cannot choose one of the $q - 1$ remaining \mathbb{F}_q -points on the conic; and we must also exclude any rational points that may lie on any of the 10 lines through pairs of the 5 points. However, only one of these 10 lines contains any rational points, and indeed it contains $q - 1$ points not on the conic. Thus as our sixth point we choose a rational point not on the line or on the conic.

We now consider the 5 new lines and 5 new conics, just as in the $a = 6$ case, and seek to find an upper bound on the number of new rational points. Only two of these five new lines are rational, and only two of the five new conics contain any new rational points; thus an upper bound on the number of rational points on the extended configuration is $6q - 3$; thus, for $q \geq 5$, one may always choose a seventh point which is rational.

For $q = 3, 4$, a computer search reveals that such collections of points exist.

$a = 4$. We first rule out $q = 2$. Note that, by Lemma 4.2, the quadratic twist of such a surface over \mathbb{F}_2 has a unique rational point. The non-existence of such a surface has been independently discovered at least twice; see [27, p. 14] and [24, Thm. 4.1.1]. Both proofs used intensive computer searches to enumerate del Pezzo surfaces over \mathbb{F}_2 and verify that no such surface exists. We give a more conceptual proof, appealing to the classification of conjugacy classes due to Urabe [34] (though of course some computation is implicitly used in the proof, as a computer was used to determine the conjugacy classes of $W(\mathbf{E}_7)$).

Lemma 4.6. *There is no del Pezzo surface S of degree 2 over \mathbb{F}_2 with $a(S) = 4$.*

Proof. An inspection of [34, Tab. 1] reveals that such a surface must have number 27, 48 or 52, in the notation of *loc. cit.*. Classes 48 and 52 have index 7, and arise by blowing up \mathbb{P}^2 in 3 rational points and either 2 closed points of degree 2 or a closed point of degree 4, respectively. The first configuration cannot arise over \mathbb{F}_2 ; indeed, after a quadratic extension, one would obtain 7 rational points in general position over \mathbb{F}_4 , which do not exist by Proposition 4.5.

For the class 52, we use conic bundles. Consider the cubic surface S' obtained by blowing-up \mathbb{P}^2 in 2 rational points P, Q and 4 points P_1, P_2, P_3, P_4 over $\bar{\mathbb{F}}_q$

which form a closed point of degree 4. This surface contains 5 lines. Two lines come from the exceptional curves of the blow-up, two are from the conics through P, P_1, P_2, P_3, P_4 and Q, P_1, P_2, P_3, P_4 , and the last (which we call L) from the line through P and Q . Notice that all the lines on S' meet L . The line L gives rise to a conic bundle $S' \rightarrow \mathbb{P}^1$ with singular fibres over 3 rational points and a closed point of degree 2. However, our surface S is obtained by blowing-up S' in a rational point not on a line. Hence it has a conic bundle $S \rightarrow \mathbb{P}^1$ with singular fibres over 4 rational points in \mathbb{P}^1 . This cannot occur over \mathbb{F}_2 as $\#\mathbb{P}^1(\mathbb{F}_2) = 3$.

Consider next a surface S with class 27, which has index 2 and contains 2 skew lines. By Table 7.1, the blow-down of these lines is a minimal del Pezzo surface S' of degree 4 and Picard number 2. Hence by [15, Thm. 1], the surface S' is equipped with a relatively minimal conic bundle $\pi : S' \rightarrow \mathbb{P}^1$. As $a(S') = 2$, Lemma 2.6 implies that the singular fibres of π lie over 2 distinct closed points of \mathbb{P}^1 of degree 2. However $\mathbb{P}_{\mathbb{F}_2}^1$ contains only 1 closed point of degree 2, which is a contradiction. This completes the proof. \square

We may therefore assume that $q \geq 3$. We will show that we may find three rational points and a closed point of degree 4 in general position. Arguing as in the previous cases, we find an upper bound on the number of rational points on the configuration of 15 lines and 6 conics defined by two rational points and a closed point of degree 4. Sparing the reader the intricate details, which are similar in spirit to the previous cases, the upper bound of $3q + 1$ is obtained, so for $q \geq 3$ one may choose a rational seventh point.

Remark 4.7. In [27, p. 14], [24, Thm. 4.1.1] an intensive computer search is also used to show that there is no del Pezzo surface S of degree 2 over \mathbb{F}_4 with a unique rational point. We are also able give an alternative proof: The quadratic twist S_σ of such a surface has $a(S_\sigma) = 6$, which we have already shown not to exist over \mathbb{F}_4 .

$a = 3$. Arguing as in the previous cases, one may compute exactly the number of rational points on the configuration of lines and conics through two rational points, and one closed point of degree 5, and obtain the answer of $q + 2$; thus this a value arises for every q .

$a = 2$. Consider the closed point of degree 6 in general position that was constructed in the $a = 1$ cubic case, and consider the 15 lines and six conics that it defines. This configuration has at most one rational point (three of the lines might cross in one point, which would then force that point to be rational; this would in fact give an Eckhardt point on the cubic surface) and so one can always choose a rational point not on this configuration.

$a = 1$. A surface with the required trace was already constructed when handling the $a = 0$ case of cubic surfaces.

On applying Lemma 4.2, this completes the proof of Theorem 1.4. \square

5. DEL PEZZO SURFACES OF DEGREE 1

We now prove Theorem 1.5. This is by far the most difficult case due to the requirement of working with the assumption that no 8 points lie on a cubic with a singularity at one of the points.

5.1. Definitions and basic properties. Let k be a field. Any del Pezzo surface S of degree 1 over k can be written in the form

$$w^2 + f_1(x, y)zw + f_3(x, y)w = z^3 + f_2(x, y)z^2 + f_4(x, y)z + f_6(x, y) \subset \mathbb{P}(1, 1, 2, 3)$$

where $\deg f_i = i$. The double of the anticanonical map is given by $(x : y : z : w) \mapsto (x : y : z)$, and realises S as a double cover of $\mathbb{P}(1, 1, 2)$. This map is separable and induces an involution of S , called the *Bertini involution*.

5.1.1. Quadratic twists. As in §4.1.2, we may perform a quadratic twist by an element of $\alpha \in H^1(k, \mathbb{Z}/2\mathbb{Z})$. Over a finite field, we denote the unique non-trivial quadratic twist of S by S_σ . We have the following analogue of Lemma 4.2.

Lemma 5.1. *Let S be a del Pezzo surface of degree 1 over a finite field \mathbb{F}_q and S_σ its non-trivial quadratic twist. Then*

$$a(S) + a(S_\sigma) = 2.$$

Proof. A similar strategy to the proof of Lemma 4.2 shows that

$$\#S(\mathbb{F}_q) + \#S_\sigma(\mathbb{F}_q) = 2\#\mathbb{P}(1, 1, 2)(\mathbb{F}_q).$$

However $\mathbb{P}(1, 1, 2)$ has $1 + q + q^2$ rational points, and the result follows. \square

5.2. Proof of Theorem 1.5. We prove Theorem 1.5 using a similar strategy to the previous cases. However, to avoid working directly with the condition in Definition 2.3 concerning cubic curves passing through the 8 points, we shall often take the following approach.

Let S' be a del Pezzo surface of degree 2. The blow-up S of S' in a rational point P is a del Pezzo surface of degree 1 if and only if P does not lie on a line nor on the ramification divisor [27, Cor. 14]. We can therefore use our constructions for del Pezzo surfaces of degree 2 to obtain certain traces, and then blow-up such a rational point to obtain the required del Pezzo surface of degree 1. The Hasse-Weil bound is used to bound the number of rational points on the lines and the ramification divisor on S' , which then guarantees the existence of such a rational point P for sufficiently large q . From Lemma 4.1, we know that the ramification divisor has at most $q + 1 + 6\sqrt{q}$ rational points for q odd, and at most $2(q + 1)$ rational points for q even. In the sequel we shall refer to the union of the lines and the ramification curve on S' as the *bad locus*.

As we appeal to the Hasse-Weil bounds, this strategy will almost never give sharp lower bounds for those q for which the required configurations of points exist. For the remaining small q , we use a range of tricks, together with a computer (with the algorithm based on Lemma 2.5) to help finish it off. Note that, when the required points in general position exist, one finds them very quickly, as a “randomly” chosen collection of points will lie in general position. For very small q (e.g. $q < 5$) we can often rule out trace values by pure thought, but for slightly higher values (e.g. $5 \leq q \leq 9$), the value of q is still small enough to

prove non-existence of points in general position on a computer, by enumerating all possibilities. (The most computationally intensive case was searching for 6 rational points and one closed point of degree 2 for $q = 9$, which took about 90 minutes to run on a desktop computer; most of the cases took less than 15 minutes). Note that, at no point in our arguments do we need to enumerate *all* del Pezzo surfaces of degree 1 in order to verify the non-existence of certain traces; our approach of enumerating configurations of closed points in \mathbb{P}^2 in general position and appealing to the classification of conjugacy classes due to Urabe [34] is substantially faster.

For the value $a = 1$, we explicitly construct a closed point in \mathbb{P}^2 of degree 8 in general position.

$a = 9$. Every such degree 1 del Pezzo surface is split.

Lemma 5.2. *Let $q \leq 13$ or $q = 17$. Then there does not exist a split del Pezzo surface of degree 1 over \mathbb{F}_q .*

Proof. Let S be a such a del Pezzo surface. The blow-down of a line L on S is a split del Pezzo surface S' of degree 2 with a rational point outside the bad locus. Hence it suffices to show that there exists no such surface S' , for q as in the statement of the lemma.

First note that we have $q \geq 9$ by Theorem 1.4. Kaplan [17, §4.3] has determined the isomorphism classes of split del Pezzo surfaces of degree 2 over the remaining relevant finite fields. As observed in [19, Cor. 4.4], this classification implies that any split del Pezzo surface of degree 2 with $q = 9, 11, 13$ has all its rational points on its lines (i.e. is “full” in the terminology of *loc. cit.*). By [17, Prop. 73] there are 7 isomorphism classes of split del Pezzo surfaces of degree 2 over \mathbb{F}_{17} , and by [19, Thm. 4.5] all but one of these is full. As explained in the proof of [19, Thm. 4.5], the non-full surface S' is branched over the Kuwata quartic curve C_{234} (see [19, §3] and [22, §8] for notation). One finds that S' has the equation

$$w^2 = x^4 + 13x^2y^2 + 13y^4 + 13x^2z^2 + 4y^2z^2 + 8z^4.$$

Using the explicit description of the lines given in [19, Thm. 3.2] (see also [22, Thm. 8.2]), a calculation shows that S' contains exactly 2 rational points which do not lie on a line. These are the points $(0 : 1 : 3 : 0)$ and $(0 : 1 : 14 : 0)$, which clearly lie on the ramification locus. This completes the proof. \square

Let $q \geq 16$, and let S be a split degree 2 del Pezzo surface over \mathbb{F}_q . This surface has all 56 of its lines rational; thus $56(q+1) + \max\{2(q+1), q+1+6\sqrt{q}\}$ is an upper bound on the rational points contained in the bad locus. Comparing this with $q^2 + 8q + 1$, the number of rational points on S , we find that there is a *good* rational point provided $q \geq 53$.

For $q = 16$ and $19 \leq q \leq 49$, after a computer search we find 8 rational points in general position in \mathbb{P}^2 .

$a = 7$. By [34, Tab. 2], every such degree 1 del Pezzo surface is the blow-up of \mathbb{P}^2 in 6 rational points and a closed point of degree 2. It follows from Theorem 1.4, that these surfaces do not exist for $q = 2, 3, 4$. A computer search enumerating such collections of points rules out the remaining values $q = 5, 7, 8, 9$.

Let $q \geq 11$, and let S be a degree 2 del Pezzo surface over \mathbb{F}_q with trace value 6 as constructed in the proof of Theorem 1.4. Being the blow-up of 5 rational points and one closed point of degree 2, S must have precisely 32 of its 56 lines rational. Moreover, if any of the remaining 24 non-rational lines have a rational point then this point must also lie on one of the rational lines. Therefore, there are at most $32(q+1) + \max\{2(q+1), q+1+6\sqrt{q}\}$ rational points in the bad locus. Comparing this with the total number q^2+6q+1 of points on S , we see that S admits a good rational point as soon as $q \geq 31$.

For $11 \leq q \leq 29$ we search for six rational points and one closed point of degree 2 lying in the plane in general position, and find them.

$a = 6$. By [34, Tab. 2], every such degree 1 del Pezzo surface S is the blow-up of \mathbb{P}^2 in 5 rational points and a closed point of degree 3. Such a surface does not exist for $q = 2, 3$, as by Lemma 5.1 its quadratic twist has $a(S_\sigma) = -4$, hence a negative number of rational points by (2.1). For $q = 4, 5$, the required configuration of points in \mathbb{P}^2 is shown not to exist after a computer search.

Let $q \geq 5$, and let S be the degree 2 del Pezzo surface over \mathbb{F}_q with trace value 5 obtained as the blow-up of \mathbb{P}^2 at 4 rational points and one closed point of degree 3. This surface has precisely 20 of its lines rational, and the rational points on the non-rational lines, if any, must also lie on the rational lines; thus $20(q+1) + \max\{2(q+1), q+1+6\sqrt{q}\}$ is a bound for the number of rational points on the bad locus which, when compared with q^2+5q+1 , shows that for $q \geq 19$, the desired degree 1 del Pezzo surface exists.

For $7 \leq q \leq 17$ the required configuration of points is found using a computer search.

$a = 5$. Here, by [34, Tab. 2], there are 4 possible conjugacy classes (numbers 37, 68, 93, 97 of *loc. cit.*). Such a surface S cannot exist when $q = 2$, as its quadratic twist would have $a(S_\sigma) = -3$, hence a negative number of rational points.

We now assume that $q \geq 4$. As showed in the previous section we may find in \mathbb{P}^2 three rational points and a closed point of degree 4 in general position. The corresponding degree 2 del Pezzo surface has trace value 4, and precisely 12 of its lines rational; in addition, precisely two of the non-rational lines meet in a rational point; thus $1+12(q+1) + \max\{2(q+1), q+1+6\sqrt{q}\}$ is an upper bound on the number of rational points in the bad locus, which is less than q^2+4q+1 provided $q \geq 13$.

A computer search shows that the required points exist for $q \geq 5$, but also reveals that there is no surface S of index 8 with $a(S) = 5$ over \mathbb{F}_q for $q \in \{2, 3, 4\}$. For $q = 3, 4$ we therefore explicitly find such a surface, necessarily not of index 8.

For $q = 3$ there is a unique such surface up to isomorphism, as discovered by Li [24, Thm. 3.1.3]. It has the equation

$$w^2 = z^3 + (2x^4 + x^2y^2 + 2y^4)z + (x^6 + 2x^4y^2 + y^6).$$

For $q = 4$, an example of a surface of trace 5 is

$$w^2 + xzw + (x^3 + u^2x^2y + uxy^2 + y^3)w = \\ z^3 + (u^2x^2 + xy + y^2)x^2z + (u^2x^4 + ux^3y + x^2y^2)y^2,$$

where $u \in \mathbb{F}_4 \setminus \mathbb{F}_2$.

$a = 4$. We showed in the previous section that, for all q , we may find in \mathbb{P}^2 two rational points and a closed point of degree 5 in general position. The corresponding degree 2 del Pezzo surface has trace value 3, and precisely 6 of its lines rational; thus $8(q+1)$ is an upper bound on the number of rational points in the bad locus, which is less than $q^2 + 3q + 1$ provided $q \geq 7$. A computer search finds the relevant points in general position for $q = 4, 5$, but shows that no index 8 surfaces occur here for $q = 2, 3$.

Surfaces over \mathbb{F}_2 with a unique rational point were found by Li [24, Thm. 3.1.3]. The quadratic twist of such a surface has trace 4, an example being

$$w^2 + (x^3 + x^2y + y^3)w = z^3 + (x^4 + x^3y + y^4)z.$$

Over \mathbb{F}_3 , an example of trace 4 is

$$w^2 = z^3 + (x+y)^2z^2 + (x^2y^2 + y^4)z + (x^6 + 2x^4y^2 + x^3y^3 + x^2y^4 + y^6).$$

$a = 3$. Continuing the above program, we consider the del Pezzo surface of degree 2 given by blowing-up \mathbb{P}^2 in a rational point and a closed point of degree 6. We find that there are rational points away from the bad locus for $q \geq 3$.

An explicit surface of trace 3 over \mathbb{F}_2 is given by

$$w^2 + yzw + (x^3 + xy^2)w = z^3 + (x^4 + xy^3)z + (x^5y + x^4y^2 + x^3y^3 + x^2y^4 + y^6).$$

$a = 2$. We consider the degree 2 del Pezzo surface over \mathbb{F}_q obtained as the blow-up of one closed point of degree 5 and one closed point of degree 2. None of the 56 lines on this surface have any rational points. The ramification divisor, and hence the bad locus, has at most $\max\{2(q+1), q+1+6\sqrt{q}\}$ rational points, which is less than the total number $q^2 + 2q + 1$ of rational points provided $q \geq 3$. For $q = 2$, a computer search yielded the required configuration of points.

$a = 1$. We will prove the existence of a closed point of degree 8 in general position. Let $\alpha_1, \dots, \alpha_8$ be a normal basis of \mathbb{F}_{q^8} over \mathbb{F}_q , that is, if σ is a generator of $\text{Gal}(\mathbb{F}_{q^8}/\mathbb{F}_q)$, then $\alpha_i = \sigma^{i-1}\alpha_1$ for $i = 1, \dots, 8$. Write $P_i = [1 : \alpha_i : \alpha_i^3]$ and note that the collection $\{P_1, \dots, P_8\}$ forms a closed point. As in the proof of Theorem 1.1, an application of Lemma 2.5 shows that no 3 are collinear and no 6 lie on conic.

It thus remains to verify that the eight matrices M_i described in Part (3) of Lemma 2.5 all have trivial kernel. To verify this we will show that, for $i = 1, \dots, 8$, the matrix given by removing the last row of M_i has non-zero determinant. This determinant is readily computed to be

$$\pm \alpha_i^3 \left(2\alpha_i + \sum_{\substack{1 \leq j \leq 8 \\ j \neq i}} \alpha_j \right) \prod_{\substack{1 \leq j \leq 8 \\ j \neq i}} (\alpha_i - \alpha_j)^2 \prod_{\substack{1 \leq k < j \leq 8, \\ k, j \neq i}} (\alpha_k - \alpha_j),$$

and our claim follows from the fact that the α_i constitute a basis.

Applying Lemma 5.1 completes this case, and thereby the whole proof, of Theorem 1.5. \square

6. AN INVERSE GALOIS PROBLEM

The aim of this section is to prove Theorem 1.7 and Theorem 1.9.

6.1. Hilbert schemes. Let $d \leq 6$. We first count all *anticanonically embedded* del Pezzo surfaces, viewed as points of some Hilbert scheme, and show that an analogous limit exists. We then quotient out by the action of the automorphism group. We work with the set-up of [16, §4]. Let

$$\mathcal{X}_d = \begin{cases} \mathbb{P}_{\mathbb{Z}}^d, & \text{if } d \geq 3, \\ \mathbb{P}(1, 1, 1, 2)_{\mathbb{Z}}, & \text{if } d = 2, \\ \mathbb{P}(1, 1, 2, 3)_{\mathbb{Z}}, & \text{if } d = 1. \end{cases}$$

Denote by \mathcal{G}_d the automorphism group scheme of \mathcal{X}_d over \mathbb{Z} . Any del Pezzo surface of degree d can be anticanonically embedded into \mathcal{X}_d . Let \mathcal{H}_d denote the Hilbert scheme over \mathbb{Z} which parametrises anticanonically embedded del Pezzo surfaces of degree d . By [16, Lem. 4.1], the morphism $\mathcal{H}_d \rightarrow \text{Spec } \mathbb{Z}$ is smooth with geometrically connected fibres.

Let $\mathcal{L}_d \rightarrow \mathcal{H}_d$ be the universal family of lines of anticanonically embedded del Pezzo surfaces of degree d (see [16, §4.2]).

Lemma 6.1. *Let $d \leq 6$. The morphism $\mathcal{L}_d \rightarrow \mathcal{H}_d$ is finite étale. The generic fibre of $\mathcal{L}_d \rightarrow \mathcal{H}_d$ is irreducible over \mathbb{Q} and its Galois closure has Galois group $W(\mathbf{E}_{9-d})$.*

Proof. That it is finite étale is well-known; see for example [28, Prop. 3.6]. For the second part, standard properties of finite étale morphisms imply that it suffices to show that for any number field k , there exists a del Pezzo surface of degree d over k whose splitting field has Galois group $W(\mathbf{E}_{9-d})$.

For $d = 5$ and $d = 6$ the result follows from the classification of such surfaces (see e.g. [31, Thm. 3.1.3] and [1, Thm. 3.5], respectively). For $d = 4$ this is proved in [21, Thm. I, p. 17]. The case of cubic surfaces has been known for a long time; a modern proof can be found in [30, Thm. 8.3]. For $d = 1$, this follows from [36, Thm. 1.3] (see [36, Rem. 1.4]). The result for $d = 2$ over \mathbb{Q} is proved in [10], and a similar argument to the one given at the end of [36, §6], which we do not reproduce here, yields the result over any number field, as required. \square

The following is now a simple consequence of the version of the Chebotarev density theorem proved by Ekedahl in [9, Lem. 1.2].

Proposition 6.2. *Let C be a conjugacy class of $W(\mathbf{E}_{9-d})$. Then*

$$\lim_{q \rightarrow \infty} \frac{\#\{S \in \mathcal{H}_d(\mathbb{F}_q) : C(S) = C\}}{\#\mathcal{H}_d(\mathbb{F}_q)} = \frac{\#C}{\#W(\mathbf{E}_{9-d})}. \quad (6.1)$$

Proof. This follows immediately from Lemma 6.1 and [9, Lem. 1.2] (the key property being that the generic fibre of $\mathcal{L}_d \rightarrow \mathcal{H}_d$ is irreducible over \mathbb{Q}). \square

6.2. Proof of Theorem 1.7. The group scheme \mathcal{G}_d acts on \mathcal{H}_d in a natural way, with two elements lying in the same orbit if and only if they are isomorphic. Using this and Proposition 6.2, we obtain

$$\begin{aligned} \lim_{q \rightarrow \infty} \frac{\sum_{S \in \mathcal{S}_d(\mathbb{F}_q), C(S)=C} \frac{1}{|\text{Aut } S|}}{\sum_{S \in \mathcal{S}_d(\mathbb{F}_q)} \frac{1}{|\text{Aut } S|}} &= \lim_{q \rightarrow \infty} \frac{\sum_{S \in \mathcal{H}_d(\mathbb{F}_q)/\mathcal{G}_d(\mathbb{F}_q), C(S)=C} \frac{1}{|\text{Aut } S|}}{\sum_{S \in \mathcal{H}_d(\mathbb{F}_q)/\mathcal{G}_d(\mathbb{F}_q)} \frac{1}{|\text{Aut } S|}} \\ &= \lim_{q \rightarrow \infty} \frac{\#\{S \in \mathcal{H}_d(\mathbb{F}_q) : C(S) = C\}}{\#\mathcal{H}_d(\mathbb{F}_q)} \\ &= \frac{\#C}{\#W(\mathbf{E}_{9-d})}, \end{aligned}$$

as required. \square

6.3. Proof of Theorem 1.9. For $d \leq 3$, we shall show that the automorphism group is constant on some open subset of \mathcal{H}_d .

Lemma 6.3. *Let $d \leq 3$, let $a_3 = 1$ and $a_1 = a_2 = 2$. There exists an open subscheme $\mathcal{U}_d \subset \mathcal{H}_d$, such that $\mathcal{U}_d \rightarrow \text{Spec } \mathbb{Z}$ is surjective and*

$$S \in \mathcal{U}_d \implies |\text{Aut } S| = a_d.$$

Proof. Let $\mathcal{A}_d \rightarrow \mathcal{H}_d$ denote the relative automorphism group scheme for the universal family of del Pezzo surfaces of degree d over \mathcal{H}_d . This naturally embeds into \mathcal{G}_d as a closed subgroup scheme, hence has finite type over \mathbb{Z} . The main theorem of [20] implies that for each $x \in \text{Spec } \mathbb{Z}$, the generic fibre of $\mathcal{A}_{d, \kappa(x)} \rightarrow \mathcal{H}_{d, \kappa(x)}$ is a finite scheme of degree a_d , where $\kappa(x)$ denotes the residue field of x . Moreover, when $d = 2$ or 1 it even has 2 rational points (corresponding to the Geiser and Bertini involution, respectively). As $\mathcal{A}_d \rightarrow \mathcal{H}_d$ is of finite type, the result follows from a simple spreading out argument. \square

We now prove Theorem 1.9. Let $\mathcal{U}_d \subset \mathcal{H}_d$ and a_d be as in Lemma 6.3. As $\mathcal{H}_{d, \mathbb{F}_q}$ is geometrically integral, the Lang-Weil estimates [23] imply that in the limit (6.1), proper Zariski closed subsets are negligible. Hence applying Lemma 6.3 and Proposition 6.2, we obtain

$$\begin{aligned} \lim_{q \rightarrow \infty} \frac{\#\{S \in \mathcal{S}_d(\mathbb{F}_q) : C(S) = C\}}{\#\mathcal{S}_d(\mathbb{F}_q)} &= \lim_{q \rightarrow \infty} \frac{\#\{S \in \mathcal{H}_d(\mathbb{F}_q)/\mathcal{G}_d(\mathbb{F}_q) : C(S) = C\}}{\#(\mathcal{H}_d(\mathbb{F}_q)/\mathcal{G}_d(\mathbb{F}_q))} \\ &= \lim_{q \rightarrow \infty} \frac{\#\{S \in \mathcal{U}_d(\mathbb{F}_q)/\mathcal{G}_d(\mathbb{F}_q) : C(S) = C\}}{\#(\mathcal{U}_d(\mathbb{F}_q)/\mathcal{G}_d(\mathbb{F}_q))} \\ &= \lim_{q \rightarrow \infty} \frac{\#\{S \in \mathcal{U}_d(\mathbb{F}_q) : C(S) = C\}/a_d}{\#\mathcal{U}_d(\mathbb{F}_q)/a_d} \\ &= \lim_{q \rightarrow \infty} \frac{\#\{S \in \mathcal{H}_d(\mathbb{F}_q) : C(S) = C\}}{\#\mathcal{H}_d(\mathbb{F}_q)} \\ &= \frac{\#C}{\#W(\mathbf{E}_{9-d})}, \end{aligned}$$

as required. \square

7. A CORRECTED VERSION OF MANIN'S AND SWINNERTON-DYER'S TABLE

In this section we present our corrected version of Manin's [25, Tab. 1, p. 176] and Swinnerton-Dyer's [32, Tab. 1] tables on cubic surfaces over finite fields.

7.1. Notation. We mostly use the same notation as Manin, except for Column 8 (see §7.1.1), and a new Column 10, which is introduced to help relate to Urabe's tables on del Pezzo surfaces of degree 1 and 2 [34]. We highlight by * a mistake in Manin's table and give the correct value (see Section 7.2 for detailed explanations on these corrections).

Note that, by Corollary 1.8, every conjugacy class of $W(\mathbf{E}_6)$ arises from some smooth cubic surface over \mathbb{F}_q for all sufficiently large q . We may therefore describe the columns in terms of the geometry of such a surface S . We denote by Fr_q the Frobenius element of $\text{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$.

0. This column denotes the numbering of the relevant conjugacy class, in the notation of Frame [11] and Swinnerton-Dyer [32].
1. This is Manin's number for the conjugacy class.
2. The index $i(S)$ is defined to be the size of the largest Galois invariant collection of pairwise skew lines over $\bar{\mathbb{F}}_q$.
3. The order of an element in the conjugacy class.
4. The reciprocal measure $\mu(C)^{-1}$ of C , defined to be $\#W(\mathbf{E}_6)/\#C$.
5. This column gives the eigenvalues of Fr_q acting on $(\text{Pic } \bar{S}) \otimes \mathbb{C}$. The notation n^b means that there are exactly b eigenvalues which are primitive n th roots of unity.
6. The trace $a(S)$ of Fr_q acting on $\text{Pic } \bar{S}$.
7. The Galois cohomology group $H^1(\mathbb{F}_q, \text{Pic } \bar{S})$. We use the notation n^2 to denote the group $(\mathbb{Z}/n\mathbb{Z})^2$.
8. The orbit type of $\text{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$ on the lines of \bar{S} (see §7.1.1).
9. Information about the blow-downs of S . If S is minimal, then this column is empty. For S of index 1, we give Manin's decomposition type [25, Tab. 2] of the quartic del Pezzo surface obtained by blowing down a line. For S of index i greater than 1, we give an orbit type of some collection of i Galois invariant skew lines over $\bar{\mathbb{F}}_q$.
10. The conjugacy class of the del Pezzo surface of degree 2 in Urabe's table [34, Tab. 1] obtained by blowing-up S in a rational point not on a line (such a point exists for sufficiently large q).

We note that we have confirmed that the decompositions claimed by Manin for quartic del Pezzo surfaces in [25, Tab. 2] and [25, Tab. 3] are correct in the latest edition of his book.

7.1.1. Column 8. A significant addition to our table is a new notation to describe the orbits. Both Swinnerton-Dyer [32] and Manin [25] only wrote down the size of each orbit, and separated orbits of the same size and different "configuration". For example, for the class C_{10} they wrote $3, 6^3, 6$ to denote an orbit of length 3 and four orbits of length 6, of which three have the same configuration and the other a different configuration.

We take a slightly different approach, in order to also say which *exact* configurations these correspond to. Let $w \in W(\mathbf{E}_6)$, considered with its action on the

27 lines of a cubic surface, and let O be an orbit of w . We define the *type* of O to be the isomorphism class its graph, i.e. the graph whose vertices are the lines in O , with an edge between intersecting lines. We define the *orbit type* of w to be the collection of types of its orbits. The graphs which arise this way are quite special; by definition they all admit a vertex-transitive action of a cyclic group. Graph theorists call such graphs *Circulant graphs*.

We notate these graphs as follows. The notation n_{m_1, \dots, m_s}^b denotes b copies of the circulant graph with vertices labelled $0, \dots, n-1$, and vertex 0 being adjacent to the vertices $\pm m_1, \dots, \pm m_s \bmod n$ (this determines all the edges of the graph). As examples, we have

$$n = \text{Edgeless graph of order } n, \quad n_1 = \text{Cycle graph of order } n.$$

The following well-known graphs occur in our table.

$$\begin{aligned} 6_{1,3} &= \text{Utility graph}, & 6_{2,3} &= \text{Triangular prism}, \\ 8_{1,4} &= \text{Wagner graph}, & 10_{1,3} &= \text{5-crown graph}. \end{aligned}$$

Urabe [34] choose to separate orbits by their *characteristic sequences*, which is essentially the list of vertices which meet the vertex 0. However, different characteristic sequences may give rise to isomorphic graphs. For example, we have $9_{1,3} \cong 9_{2,3} \cong 9_{3,4}$ (in the above notation). In particular, the reader should bear in mind that some of the orbits in Urabe's table which are separated by their characteristic sequences have isomorphic graphs.

7.2. Corrections. We now explain the issues with Manin's table and how to fix them.

Typesetting. There are some typesetting issues with Columns 5 and 8 which make them difficult to read; we have fixed these formatting issues in our table.

Column 5: The mistake here is a minor typo coming from recording the incorrect value in Swinnerton-Dyer's table [32], and easily corrected.

Column 7: The errors regarding the calculation of $H^1(\mathbb{F}_q, \text{Pic } \bar{S})$ were first discovered by Urabe [34]; we have nothing new to add.

We now explain the mathematical mistakes we found, regarding the index and the orbit type. The corrected values appear here for the first time.

Column 2: Let q be a sufficiently large prime power.

C_{25} : Let S' be a smooth non-split quadric surface over \mathbb{F}_q . Let S be the cubic surface obtained by blowing-up S' in a closed point of degree 5 in general position. One easily sees that the splitting field of S has degree 10, hence such a surface must have class C_{25} . But, by construction, one may contract an orbit of length 5 on S , hence the index is not 2, as claimed by Manin.

C_8 : Again let S' be a smooth non-split quadric surface over \mathbb{F}_q . Let S be the blow-up of S' in closed points of degree 2 and 3 in general position. This surface has splitting field of degree 6 and $a(S) = 0$, thus it must have class C_8 . As before, we see that the index is not 3, as claimed by Manin.

0. Class	1. No.	2. $i(S)$	3. Order	4. $\mu(C)^{-1}$	5. Eigenvalues	6. $a(S)$	7. H^1	8. Orbit type	9. Blow down	10. Blow-up
C_{13}	1	0	12	12	$1, 3^2, 12^4$	0	0	$3_1, 12_{1,4,6}^2$		22
C_{12}	2	0	6	72	$1, 3^2, 6^4$	2	0	$3_1, 6_{2,3}^4$		24
C_{11}	3	0	3	648	$1, 3^6$	-2	3^2	3_1^9		20
C_{14}	4	0	9	9	$1, 9^6$	1	0	$9_{1,3}^3$		23
C_{10}	5	0	6	36	$1, 2^2, 3^2, 6^2$	-1	2^2	$3_1, 6_{1,3}^3, 6_{2,3}^2$		21
C_{24}	6	1	12	12	$1^2, 2, 4^2, 6^2$	2	0	$1, 4_2, 4_1, 6_3, 12_{2,3}$	I	33
C_{20}	7	1	8	8	$*1^2, 2, 8^4$	1	$*0$	$1, 2_1, 8_4, 8_{1,4}^2$	XVIII	32
C_7	8	1	6	36	$1^3, 2^2, 6^2$	2	0	$1^3, 2_1^3, 6_3^3$	II	26
C_{19}	9	1	4	96	$1^2, 2^3, 4^2$	-1	2^2	$1, 2_1^3, 4_2, 4_1^4$	X	29
C_4	10	1	4	96	$1^3, 4^4$	3	$*0$	$1^3, 4_2^6$	V	27
C_3	11	1	2	1152	$1^3, 2^4$	-1	2^2	$1^3, 2_1^{12}$	IV	25
C_{25}	12	$*5$	10	10	$1^2, 2, 5^4$	0	0	$2, 5, 5_1^2, 10_{1,3}$	5	58
C_{22}	13	3	6	36	$1^2, 2, 3^4$	-1	0	$3^2, 3_1^3, 6_1^2$	3	42
C_8	14	$*5$	6	24	$1^3, 2^2, 3^2$	0	0	$1, 2^2, 2_1^2, 3^2, 6_1^2$	2, 3	53
C_{23}	15	6	6	12	$1^2, 2, 3^2, 6^2$	1	0	$3_1, 6^2, 6_{1,3}, 6_{2,3}$	6	59
C_{15}	16	6	5	10	$1^3, 5^4$	2	0	$*1^2, 5^3, 5_1^2$	1, 5	56
C_5	17	6	4	16	$1^3, 2^2, 4^2$	1	0	$1, 2^2, 2_1, 4^2, 4_2, 4_1^2$	2, 4	55
C_9	18	6	3	108	$1^3, 3^4$	1	0	$3^6, 3_1^3$	3, 3	54
C_{18}	19	6	4	32	$1^4, 2, 4^2$	3	0	$1^5, 2_1, 4^4, 4_2$	$1^2, 4$	52
C_{21}	20	6	6	36	$1^4, 2, 3^2$	2	0	$1^3, 2^3, 3^4, 6_1$	1, 2, 3	51
C_{17}	21	6	2	96	$1^4, 2^3$	1	0	$1^3, 2^6, 2_1^6$	2^3	50
C_6	22	6	3	216	$1^5, 3^2$	4	0	$1^9, 3^6$	$1^3, 3$	49
C_2	23	6	2	192	$1^5, 2^2$	3	0	$1^7, 2^8, 2_1^2$	$1^2, 2^2$	48
C_{16}	24	6	2	1440	$1^6, 2$	5	0	$1^{15}, 2^6$	$1^4, 2$	47
C_1	25	6	1	51840	1^7	7	0	1^{27}	1^6	46

TABLE 7.1. Conjugacy classes of $W(\mathbf{E}_6)$.

These constructions show that such surfaces S have index at least 5. However, in both cases we have $a(S) = 0$, hence Lemma 2.2 implies that the index is not 6, so the index is indeed 5 as claimed.

Column 8: We found an error regarding the orbit type of class C_{15} . This mistake can be traced back to Table 1 of Swinnerton-Dyer [32]. Swinnerton-Dyer claims that the configuration is $1^2, 5^2, 5^2, 5$ (in his notation), however it is in fact $1^2, 5^3, 5_1^2$ (in our notation).

Let q be arbitrary and consider the blow-up of \mathbb{P}^2 in a rational point and a closed point of degree 5 in general position over \mathbb{F}_q . This has splitting field of degree 5, hence by Table 7.1 it corresponds to the class C_{15} . The orbits of length 1 come from the rational point and the conic passing through the quintic points. The orbits of length 5 arise as follows:

- (1) 5 lines above the quintic points.
- (2) 5 lines passing through the rational point and one of the quintic points.
- (3) 5 conics passing through the rational point and four of the quintic points.
- (4) 5 lines passing through adjacent quintic points.
- (5) 5 lines passing through non-adjacent quintic points.

One easily checks that the first three types consist of pairwise skew lines, whereas the last two types consist of lines meeting exactly two others (this can be deduced from [12, Rem. V.4.10.1], for example). This gives rise to orbit type $1^2, 5^3, 5_1^2$, as claimed.

Our guess is that Urabe over-looked the remaining mistakes in Manin's table as they concern the index and orbit type, which can behave erratically with respect to blow-ups.

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BARINDER BANWAIT, FAKULTÄT FÜR MATHEMATIK, UNIVERSITÄT DUISBURG-ESSEN, THEA-LEYMANN-STR. 9, 45127 ESSEN, GERMANY.

E-mail address: b.s.banwait86@gmail.com

FRANCESC FITÉ, FAKULTÄT FÜR MATHEMATIK, UNIVERSITÄT DUISBURG-ESSEN, THEA-LEYMANN-STR. 9, 45127 ESSEN, GERMANY.

E-mail address: francesc.fite@gmail.com

DANIEL LOUGHRAN, LEIBNIZ UNIVERSITÄT HANNOVER, INSTITUT FÜR ALGEBRA, ZAHLENTHEORIE UND DISCRETE MATHEMATIK, WELFENGARTEN 1, 30167 HANNOVER, GERMANY.

E-mail address: loughran@math.uni-hannover.de

URL: <http://www.iazd.uni-hannover.de/~loughran/>